KILLARA HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

2021 | Mathematics Extension 2

Trial HSC Examination

Date: Monday 23th August, 2021

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks

- Allow about 15 minutes for this section

Section II – 90 marks

- Allow about 2 hours and 45 minutes for this section

Section I (10 marks)	Multiple Choice	/10
Section II (90 marks)	Question 11	/15
	Question 12	/15
	Question 13	/13
	Question 14	/17
	Question 15	/15
	Question 16	/15
	Total	**/100**

This question paper must not be removed from the examination room.
This assessment task constitutes 0% of the course.

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Section I

10 marks
Allow about 15 minutes for this section

Use the multiple-choice sheet for Question 1–10

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1. If $z_1 = \sqrt{3} + i\sqrt{3}$ and $z_2 = \sqrt{3} + i,$ then the complex number $\left(\frac{z_1}{z_2}\right)$ lies in the quadrant

A. I

B. II

C. III

D. IV

2. The resolved part of the force $\underset{\sim}{F} = \underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$ in the direction of $\underset{\sim}{a} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 4\underset{\sim}{k}$ is:

A. $\frac{13}{9}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

B. $\frac{13}{9}\left(\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

C. $\frac{13}{9}\left(-\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

D. $\frac{26}{3}\left(\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

3. The direction cosines of a line are $\left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$, then

A. $c > 0$

B. $0 < c < 1$

C. $c = \pm\sqrt{3}$

D. $c > 2$

Section I continues

4. The period of the function $f(x) = |\sin x| + |\cos x|$ is:

A. $\frac{\pi}{2}$

B. π

C. $\frac{3\pi}{2}$

D. 2π

5. The solutions to $z^n = 1 + i, \quad n \in \mathcal{Z}^+$ are given by

A. $2^{\frac{1}{2n}} e^{i\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right)}, \;\; k \in \mathcal{R}$

B. $2^{\frac{1}{n}} e^{i\left(\frac{\pi}{4n} + 2\pi k\right)}, \;\; k \in \mathcal{Z}$

C. $2^{\frac{1}{2n}} e^{i\left(\frac{\pi}{4} + \frac{2\pi k}{n}\right)}, \;\; k \in \mathcal{Z}$

D. $2^{\frac{1}{2n}} e^{i\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right)}, \;\; k \in \mathcal{Z}$

6. If $f(y) = e^y, \quad g(y) = y\,; \quad y \geq 0$ and

$$F(t) = \int_0^t f(t-y)g(y)dy, \qquad \text{then}$$

A. $F(t) = t\, e^{-t}$

B. $F(t) = 1 - e^{-t}(1 + t)$

C. $F(t) = e^t - (1 + t)$

D. $F(t) = t\, e^{-t}$

Section I continues

7. Which of the following statements is a negation of the following statement?

$$x \in \mathcal{R} \Leftrightarrow \ |x| \geq 0$$

A. $\{x \in \mathcal{R}^+ \text{ and } |x| > 0\}$ and $\{|x| > 0 \text{ and } x \in \mathcal{R}\}$

B. $\{x \in \mathcal{R} \text{ and } |x| \ngtr 0\}$ and $\{|x| \geq 0 \text{ and } x \notin \mathcal{R}\}$

C. $\{x \in \mathcal{R} \text{ and } |x| \ngeq 0\}$ or $\{|x| \geq 0 \text{ and } x \notin \mathcal{R}\}$

D. $\{x \in \mathcal{R}^+ \text{ and } |x| \ngtr 0\}$ and $\{|x| < 0 \text{ and } x \notin \mathcal{R}\}$

8. A particle of mass m moves in a straight line under the action of a resultant force F where $F = F(x)$.

Given that the velocity v is v_0 when the position is $x = x_0$, and that v is v_1 when x is x_1, it follows that $|v_1| =$

A. $\sqrt{\frac{2}{m}}\int_{x_0}^{x_1} \sqrt{F(x)}dx + v_0$

B. $\sqrt{2}\int_{\sqrt{x_0}}^{\sqrt{x_1}} F(x)dx + v_0$

C. $\sqrt{\frac{2}{m}\int_{x_0}^{x_1} F(x)dx + (v_0)^2}$

D. $\sqrt{\frac{2}{m}\int_{x_0}^{x_1} [F(x) + (v_0)^2]dx}$

Section I continues

9. If a, b and c are real numbers such that $a^2 + b^2 + c^2 = 1$, then $ab + bc + ca$ lies in the interval

A. $\left[-\frac{1}{2}, 1\right]$

B. $\left[\frac{1}{2}, 2\right]$

C. $\left[-1, \frac{1}{2}\right]$

D. $[-1, 2]$

10 Grin could correctly write the value of

$$\binom{15}{0} + 3\binom{15}{1} + 5\binom{15}{2} + \cdots + (2n+1)\binom{15}{n} + \cdots + 31\binom{15}{15}.$$

What did he write down?

A. 2^{15}

B. 2^{16}

C. 2^{19}

D. 2^{31}

End of Section I

Section II

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your response should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Given $z = 1 - i$, find:

(i) $Im\left(\frac{1}{z}\right)$ **2**

(ii) z^{10} in cartesian form **2**

(iii) The two values for ω such that $\omega^2 = 3\bar{z} + i$. **3**

(b) If $2 - 3i$ is a zero of the polynomial $z^3 + pz + q$ where p and q are real, find the values of p and q. **3**

(c) Find **3**

$$\int \frac{x^2 + 1}{x^4 + x^2 + 1}\,dx$$

(d) Find **2**

$$\int \frac{dx}{\sqrt{3 + 4x - 4x^2}}$$

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Use the substitution $\mathrm{t} = \tan\frac{\theta}{2}$ to find **3**

$$\int \frac{\tan\theta}{1+\cos\theta}\,d\theta$$

(b) Consider the propositions

$$p\text{:}\ \ 2^n - 1\ \text{ is prime} \qquad\qquad q\text{:}\ \ n \text{ is prime,} \qquad \mathrm{n} \in \mathcal{Z}^+$$

(i) Give an indirect proof by contrapositive of the following proposition: **3**

$$\text{If } \ 2^n - 1 \ \text{ is prime then } n \ \text{ is prime}$$

(ii) Write down the converse of the proposition in (i) **1**

(iii) Is the converse of the proposition in (i) true? **1**

If yes, give a proof.

If no, give reasons.

(c) (i) Find real numbers A, B and C such that **3**

$$\frac{9}{(2x-1)(x+1)^2} = \frac{A}{2x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

(ii) Hence, evaluate **2**

$$\int_1^2 \frac{9}{(2x-1)(x+1)^2}\,dx$$

(d) The complex number z is such that $|z-1| \leq 1$ and $|z-2| = 1$. Find the maximum possible value of $|z|^2$. **2**

End of Question 12

Question 13 (13 marks) Use the Question 13 Writing Booklet.

(a) Using principle of mathematical induction, prove that $\forall n > 5,\ n \in \mathcal{N}$, **3**

$$n! < \left(\frac{n}{2}\right)^n$$

(b) (i) $I_n = \int x^n e^{ax} dx$, where a is a constant. **2**

$$\text{Prove that } I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}.$$

(ii) Hence, find the value of **3**

$$\int_0^1 x^3 e^{2x} dx$$

(c) (i) Show that $x = r\cos(\omega t + \phi)$, where r, ω, ϕ are constants, is a solution of **1**

$$\frac{d^2x}{dt^2} = -\omega^2 x.$$

A small naval target rises and falls with SHM of a period of 10 seconds. The height of the waves from the crest to the trough is 2 metres.

At a horizontal range of 2000 m, a gun is fired so that the target would be hit provided it remains stationary in its highest position. The horizontal component of velocity is $1000\ ms^{-1}$.

(ii) Show that the target would be missed by a vertical height of 0.69 m. **4**

End of Question 13

Question 14 (17 marks) Use the Question 14 Writing Booklet.

(a) A vehicle of mass m kg moves in a straight line subject to a resistance $P + Qv^2$ Newtons, where v is the speed and P and Q are constants with $Q > 0$.

(i) Form an equation of motion for the acceleration of the vehicle. **1**

(ii) Hence, show that if $P = 0$, the distance required to slow down from the speed $\frac{3U}{2}$ to speed U is $\frac{\mathrm{m}}{\mathrm{Q}} \ln\left(\frac{3}{2}\right)$. **3**

(iii) Also show that if $P > 0$, the distance required to stop from speed U is given by **3**

$$D = \lambda \ln(1 + kU^2)$$

Where k and λ are constants.

(b) Let A, B, C and D represent four complex numbers z_1, z_2, z_3 and z_4 such that

$$|z_1| = |z_2| = |z_3| = |z_4| = 1 \qquad \text{and} \qquad z_1 + z_2 + z_3 + z_4 = 0.$$

(i) Prove that **2**

$$|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2 + |z_4 - z_1|^2 \geq 8$$

(ii) α. Using (i), or otherwise, prove that **1**

$$|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2 + |z_4 - z_1|^2 = 2(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)$$

if and only if $z_1 + z_3 = 0 \quad and \;\; z_2 + z_4 = 0$.

β. Draw an Argand diagram to illustrate this result, and explain its geometrical significance. **2**

Question 14 continues

Question 14 (continued)

(iii) Consider two complex numbers u and v defined as $|u| = |v| = 1$ and $uv \neq -1$. ω is a complex number such that $\omega = \dfrac{u - v}{1 + uv}$, **2**

Prove that $Re(\omega) = 0$.

(iv) If $|u| < 1$, $|v| < 1$ and $uv \neq -1$, and $\xi = \dfrac{u - v}{1 + \bar{u}\, v}$, prove that **3**

$$|\xi| \geq \frac{\big||u| - |v|\big|}{1 - |u||v|}$$

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

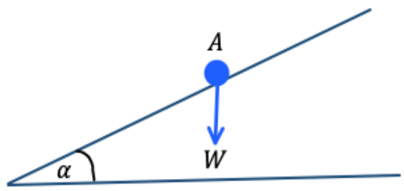
(a) (i) Prove for every positive integer k, **2**

$$\frac{1}{k^2} < \frac{1}{k-\frac{1}{2}} - \frac{1}{k+\frac{1}{2}}$$

(ii) Hence, deduce that if n is a positive integer, then **3**

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2$$

(b) The diagram below shows a plane inclined at an angle α to the horizontal. A body of weight W can be supported on this plane independently by either the force P that is acting parallel to the incline plane, or Q that is acting parallel to the horizontal plane. **2**



By drawing free body diagrams in each case, Prove that

$$\frac{1}{P^2} - \frac{1}{Q^2} = \frac{1}{W^2}$$

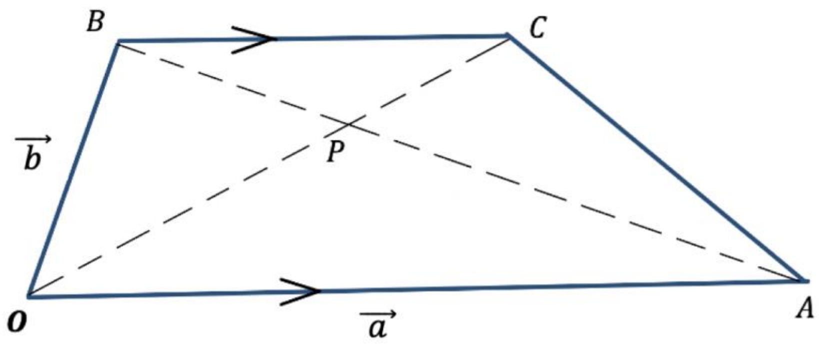
(c) Find the shortest distance from the point $P(1,2,0)$ to the line with parametric equation **3**

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda \in \mathcal{R}.$$

Question 15 continues

Question 15 (continued)

(d) In the diagram below, $OACB$ is a trapezium, with OA and OB represented by vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ respectively. The diagonals of the trapezium intersect at the point P as shown in the diagram.



(i) Show that the vector equation of the line OC is **1**

$$\overrightarrow{r} = t\left(\overrightarrow{b} + \alpha\overrightarrow{a}\right), \qquad t, \alpha \in \mathcal{R}.$$

(ii) Prove by vector methods that the point of intersection of the diagonals of a trapezium lies on the line passing through the midpoints of the parallel sides. **4**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) Prove that if a real number c satisfies a polynomial equation of the form **4**

$$r_3x^3 + r_2x^2 + r_1x + r_0 = 0,$$

where, r_0, r_1, r_2 and r_3 are rational numbers, then c satisfies an equation of the form

$$n_3x^3 + n_2x^2 + n_1x + n_0 = 0,$$

where n_0, n_1, n_2 and n_3 are integers.

(b) Consider the compound function **2**

$$f\big(x(\theta)\big) = h + x(\theta)tan\theta - \frac{g}{2v^2}\big(x(\theta)\big)^2 \sec^2\theta,$$

where g, h and v are constants and $0 < \theta < \frac{\pi}{2}$.

Using implicit differentiation or otherwise, Prove that the value of θ that maximises $x(\theta)$ can be given by

$$x_m = \frac{v^2}{g}\cot\theta_m$$

NOTE: θ_m is the angle that gives the maximum value x_m

(c) A projectile of unit mass launched from a tower of height h metres with velocity $v\ m/s$ at an angle of θ.

The equations of motion of the particle are as shown below:

$$\underset{\sim}{r} = \begin{pmatrix} v\cos\theta t \\ h + v\,sin\theta t - \frac{1}{2}gt^2 \end{pmatrix}$$

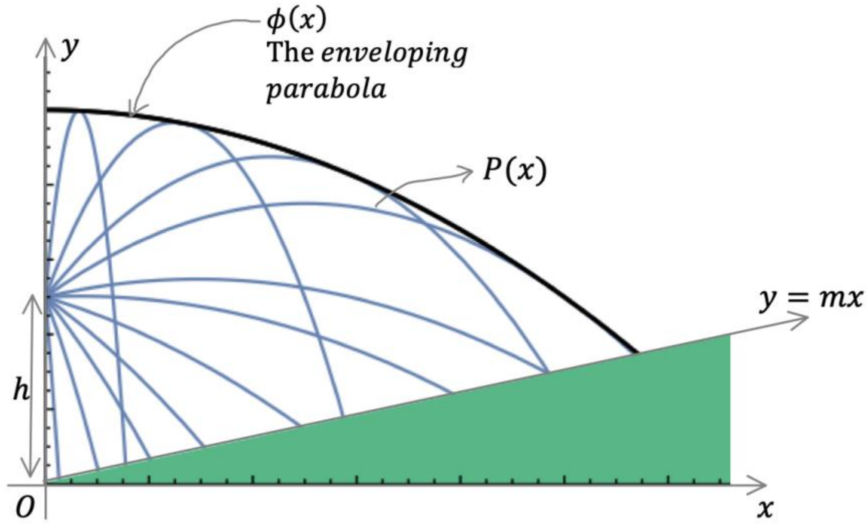
DO NOT PROVE THIS

(i) Show that, for a given launch velocity v and angle θ, the Cartesian equation of the trajectory of the particle is given by **1**

$$y = P(x) = h + xtan\theta - \frac{gx^2}{2v^2}\sec^2\theta$$

Question 16 continues

The diagram below shows the trajectory of the projectile launched at velocity $v\ m/s$ for varying values of θ. The function $\phi(x)$ is the enveloping parabola that encloses all possible paths, including the highest and the longest.



(ii) For any **given value of x** within the range of the projectile and launch velocity v, show that the equation of the enveloping parabola is **3**

$$y = h + \frac{v^2}{2g} - \frac{gx^2}{2v^2}$$

(HINT: you may set $u = \tan\theta$)

(iii) The projectile impacts on an inclined plane with equation $y = mx$ at the point $x = c$ as shown in the diagram above. **2**
Show that the x- coordinate of the point of intersection of the projectile and the inclined plane is given by

$$c = -\frac{mv^2}{g} \pm \frac{v^2}{g}\sqrt{m^2 + \frac{2gh}{v^2} + 1}$$

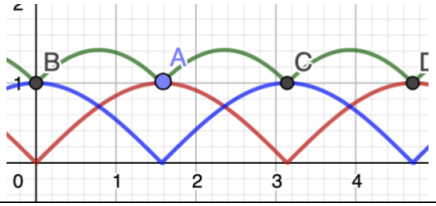
(iv) Hence, show that the optimal angle θ_m that gives the largest range is given by **3**

$$\theta_m = \cot^{-1}\left(-m + \sqrt{m^2 + \frac{2gh}{v^2} + 1}\right)$$

(You may refer to the result in (b).)

End of Examination

2021 Ext 2 Trial Marking scheme

No:	Sample Answer	Marking scheme	Feedback
1	A $Arg\ z_1 = \frac{\pi}{4}$ 1st quadrant $Arg\ z_2 = \frac{\pi}{6}$ 1st quadrant $Arg\frac{z_1}{z_2} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$ 1st quadrant Also, $Arg\ z_2 < Arg\ z_1$		
2	B $\frac{F.v}{v.v}v = \frac{2+8+16}{4+16+16}\begin{pmatrix}2\\4\\-4\end{pmatrix} = \frac{13}{9}\begin{pmatrix}1\\2\\-2\end{pmatrix}$		
3	C $\left(\frac{1}{c}\right)^2 + \left(\frac{1}{c}\right)^2 + \left(\frac{1}{c}\right)^2 = 1$ $\frac{3}{c^2} = 1 \rightarrow \quad c^2 = 3 \rightarrow \quad c = \pm\sqrt{3}$	Sum of squares of direction cosines is 1.	
4	A 		
5	D $z^n = \sqrt{2}\,e^{i\left(\frac{\pi}{4}+2\pi k\right)}, \quad k \in \mathcal{Z}$ $z = 2^{\frac{1}{2n}}e^{i\left(\frac{\pi}{4n}+\frac{2\pi k}{n}\right)}, \quad k \in \mathcal{Z}$		
6	B $F(t) = \int_0^1 f(t-y)g(y)dy$		

	$= \int_0^t f e^{t-y}\, y\, dy$ Let $t - y = p$ $-dy = dp;$ $Also,\ \ y = 0,\ \ p = t;\quad \mathrm{y = t\ ,\ p = 0}$ $\therefore F(t) = -\int_1^0 (t-p)e^p dp$ $= \int_0^1 t\ e^p dp - \int_0^1 p\ e^p dp$ $= t[e^p]_0^1 - \ [pe^p]_0^1 + \ [e^p]_0^1$ $= [t(e^t - 1) - te^1 + (e^1 - 1)]$ $F(t) = e^t - (t+1)$		
7	C Here $p: x \in \mathcal{R}\ \ \ and\ \ \ q: \|x\| \geq 0$ $p \Leftrightarrow q$ is a biconditional statement. $\neg(p \Leftrightarrow q) = \neg(p \Rightarrow q\ and\ q \Rightarrow p)$ $= \neg(p \Rightarrow q)\quad or\ \ \neg(q \Rightarrow p)$ $= \ \ (p\ and\ \neg q)\quad or\ \ \ (q\ and\ \neg p)$ Solution: $\{x \in \mathcal{R}$ and $\|x\| \ngeq 0\}$ or $\{\|x\| \geq 0$ and $x \notin \mathcal{R}\}$		

8	C $m\ddot{x} = F(x)$ $\ddot{x} = \dfrac{F(x)}{m}$ $v\dfrac{dv}{dx} = \dfrac{1}{m}F(x)$ $\int_{v_0}^{v_1} v\,dv = \dfrac{1}{m}\int_{x_0}^{x_1} F(x)\,dx$ $\dfrac{1}{2}((v_1)^2 - (v_0)^2) = \dfrac{1}{m}\int_{x_0}^{x_1} F(x)\,dx$ $(v_1)^2 = \dfrac{2}{m}\int_{x_0}^{x_1} F(x)\,dx + (v_0)^2$ $v_1 = \sqrt{\dfrac{2}{m}\int_{x_0}^{x_1} F(x)\,dx + (v_0)^2}$		
9.	A Let $ab + bc + ca = k$ Now, $(a+b+c)^2 = a^2 + b^2 + c^2 + 2k$ $(a+b+c)^2 \geq 0 \quad perfect\ square$ $1 + 2k \geq 0 \implies k \geq -\dfrac{1}{2}$ Consider $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$ $2[a^2 + b^2 + c^2 - ab - bc - ca] \geq 0$ $1 - k \geq 0 \quad \rightarrow \quad k \leq 1$ $\left[-\dfrac{1}{2}, 1\right]$		
10	C $I = \binom{15}{0} + 3\binom{15}{1} + 5\binom{15}{2} + \cdots + (2n+1)\binom{15}{n} + \cdots + 31\binom{15}{15}.$ $J = 31\binom{15}{15} + 29\binom{15}{14} + 27\binom{15}{13} + \cdots + (31-2n)\binom{15}{15-n} + \cdots + \binom{15}{0}.$ J is the same as I, but the terms are arranged in the reverse order as $\binom{n}{r} = \binom{n}{n-r}.$ Hence, $\binom{15}{0} = \binom{15}{15}; \quad \binom{15}{1} = \binom{15}{14} \ldots\ldots$		

	$I + J = 32\left[\binom{15}{0} + \binom{15}{1} + \binom{15}{2} + \cdots + \binom{15}{n} + \cdots + \binom{15}{15}\right].$ $2I = 32 \times 2^{15} = 2^{18}$		

Question 11

a)(i)	$z = 1 - i$ $\frac{1}{z} = \frac{1}{1-i} = \frac{1+i}{2}$ $Im\left(\frac{1}{z}\right) = \frac{1}{2}$	1 mark: multiplies by the conjugate 1 mark: gives the imaginary component	
a(ii)	$z^{10} = \left(\sqrt{2}e^{-\frac{i\pi}{4}}\right)^{10}$ $= 32\, e^{-\frac{10\pi}{4}} = \quad 32\, e^{-\frac{\pi}{2}} = -32i$	1 mark: expresses z^{10} in component form of exponential form. 1 mark: correctly evaluates and gives the answer in Cartesian form.	
a(iii)	$\omega^2 = 3\bar{z} + i = 3(1+i) + i$ $\omega^2 = 3 + 4i = (a+ib)^2$ $a^2 - b^2 = 3$ $a^2 + b^2 = \sqrt{(a^2-b^2)^2 + (2ab)^2}$ $a^2 + b^2 = \sqrt{3^2 + 4^2} = 5$ $Hence, 2a^2 = 8 \rightarrow \quad a = \pm 2$ $Hence, 2b^2 = 2 \rightarrow \quad b = \pm 1$ $Thus\ \ \omega^2 = \sqrt{(2+i)^2} = \pm(2+i)$	3 marks: Correct answer from correct working 2 marks: Equates the real and imaginary parts and writes the relationship attempting to find a and b, writes the squareroot from their a and b 1 mark: Equates the real and imaginary parts, but equations or method incorrect	
b)	$z^3 + pz + q = 0$ $2 - 3i$ is a root	1 mark: substitutes into the polynomial	

	$(2-3i)^3+(2-3i)p+q=0$ $\mathbf{1\ mark}$ $8-36i-54+27i+(2-3i)p+q=0$ $-46-9i+2p+q-3ip=0$ $2p+q=46 \quad 3p=\ -9;\quad 1\ mark$ $p=\ -3$ $\therefore\quad q=46-2\times -3=52\ \ 1\ mark$ Or Let the roots be $\alpha,\bar{\alpha},\beta$ Sum of roots is zero. $(2-3i)+(2+3i)+\beta=0$ $\therefore \beta=\ -4$ Product of roots = $(2-3i)(2+3i)(-4)=\ -q$ $\therefore\quad q=52$ $p=\ (2-3i)(-4)+(2+3i)(-4)+\ (2-3i)(2+3i)$ $\quad=\ -3$	1 mark: equates the real and imaginary parts 1 mark: solves for p and q correctly	
c)	$$\int\frac{x^2+1}{x^4+x^2+1}dx$$ $$\int\frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx$$ $$let\ u=x-\frac{1}{x}$$ $$du=1+\frac{1}{x^2}dx\quad (give\ 1\ mark\ here)$$ $$\int\frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx=\ \int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3}dx\quad 2\ mark$$	2 mark: reorganises the integrand, chooses the correct substitution, expresses dx and the integrand into an integrable form 1 mark: minor error or significant progress with relevant working	

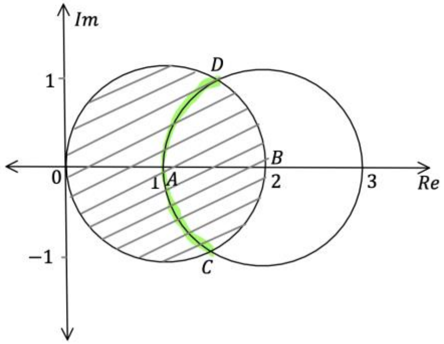
	$$\int \frac{du}{u^2+3} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + C$$ $$\frac{1}{\sqrt{3}}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{3}} + C$$ $$\frac{1}{\sqrt{3}}\tan^{-1}\frac{x^2-1}{\sqrt{3}x} + C$$	1 mark: correctly integrates and expresses the solution in terms of x	
d)	$$\int \frac{dx}{\sqrt{3+4x-4x^2}} =$$ Since, $3+4x-4x^2 \;=\; 4\left(\frac{3}{4}+x-x^2\right)$ $$4\left(1-\left(x-\frac{1}{2}\right)^2\right)$$ $$\int \frac{dx}{\sqrt{4\left(1-\left(x-\frac{1}{2}\right)^2\right)}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-\left(x-\frac{1}{2}\right)^2}}$$ $$Let \quad u = x-\frac{1}{2}$$ $$du = dx$$ $$\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u = \frac{1}{2}\sin^{-1}\left(x-\frac{1}{2}\right) + C$$ $$\frac{1}{2}\sin^{-1}\left(\frac{2x-1}{2}\right) + C$$	1 mark: completes the square and expresses the integrand in the $\frac{dx}{\sqrt{1-\left(x-\frac{1}{2}\right)^2}}$ form 1 mark: correctly integrates and expresses the answer in terms of x.	

Question 12

a)	$Let\ t = \tan\frac{\theta}{2}, then$ $tan\theta = \frac{2t}{1-t^2}, cos\theta = \frac{1-t^2}{1+t^2}\ and\ \frac{d\theta}{dt} = \frac{2}{1+t^2}$ $\int \frac{\tan\theta}{1+\cos\theta} d\theta = \int \frac{2t}{1-t^2} \div \left(1 + \frac{1-t^2}{1+t^2}\right) \times \frac{2}{1+t^2} dt$ $= \int \frac{2t}{1-t^2} \times \left(\frac{1+t^2}{2}\right) \times \frac{2}{1+t^2} dt$ $= \int \frac{2t}{1-t^2}\ dt$ $= -\ln\|1-t^2\| + C$ $= -\ln\left\|1 - \tan^2\frac{\theta}{2}\right\| + C$	3 mark: correct solution showing all working 2 mark: correctly converts the integrand, but error in simplification or integration 1 mark: correct conversion of integrand Or correct integration of the integrand in t.	
b)(i)	p: $2^n - 1$ is prime $\quad q$: n is prime We need to prove $p \Longrightarrow q$ We have $p \Longrightarrow q \equiv \neg q \Longrightarrow \neg p$ We'll prove the contrapositive statement $\neg q \Longrightarrow \neg p$ Proof: $\neg q$: n is composite. 1 mark Then $\exists\ a, b \in \mathcal{Z}^+ : n = ab,\ a \neq 1,\ b \neq 1.$ Hence, $2^{ab} - 1 = (2^a)^b - 1$ $= (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + 2^{a(b-3)} + \cdots .. + 1)$ Where $2^a - 1 \neq 1, since\ a \neq 1\ and$	* writes the contrapositive statement correctly *expresses $2^a - 1$ with n as a composite number *clearly states , $b \in \mathcal{Z}^+ : n = ab,\ a \neq 1,\ b \neq 1.$ *expresses $2^n - 1$ in factored form * Clearly explains $2^a - 1 \neq 1$ *States $2^a - 1 \neq 2^{ab} - 1$ $since\ \ b \neq 1$	

	$2^a - 1 \neq 2^{ab} - 1 \; since \;\; b \neq 1$ So, $2^n - 1$ is composite, $\neg p$ is true. We have proved $\neg q \implies \neg p$ Hence, $p \implies q$ is true.	*states the contrapositive statement and hence the given proposition is true. 3 marks: All 7 components are clearly demonstrates 2 mark: at least five 1 mark: at least 3	
b(ii)	The converse of $p \implies q$ is $q \implies p$. If n is prime, then $2^n - 1$ is prime.	1 mark:	
b(iii)	The converse is false. Let $n = 11$, $2^{11} - 1 = 23 \times 89$ is composite, thus disproving $q \implies p$	1 mark: reasons correctly	
c)(i)	$\frac{9}{(2x-1)(x+1)^2} = \frac{A}{2x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ $9 = A(x+1)^2 + B(2x-1)(x+1) + C(2x-1)$ $x = -1, \quad 9 = -3C \implies C = -3$ $x = \frac{1}{2}, \quad 9 = \frac{9}{4}A \implies A = 4$ Compare constants, $9 = A - B - C \implies B = -2$ $\frac{9}{(2x-1)(x+1)^2} = \frac{4}{2x-1} - \frac{2}{x+1} - \frac{3}{(x+1)^2}$ Or $x = \frac{1}{2}, \quad A = \frac{9}{\left(\frac{3}{2}\right)^2} = 4$ $x = -1, \quad C = \frac{9}{2(-1)-1} = -3$	3 marks: All pronumerals are calculated correctly showing all working 2 marks: two of the pronumerals are calculated correctly showing all working 1 mark: A valid method is used to evaluate the constants.	

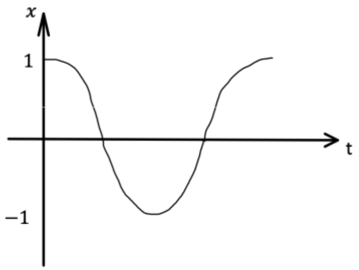
	Let $x = 0$, then $\frac{9}{-1 \times -1} = \frac{4}{-1} + B - 3$ $\therefore B = -2$		
c)(ii)	$\int_1^2 \frac{9}{(2x-1)(x+1)^2} dx =$ $\int_1^2 \frac{4}{2x-1} - \frac{2}{x+1} - \frac{3}{(x+1)^2} dx$ $\left[2\ln\lvert 2x-1\rvert - 2\ln\lvert x+1\rvert - \frac{3(x+1)^{-1}}{-1}\right]_1^2$ $= \left[2\ln\left\lvert\frac{2x-1}{x+1}\right\rvert + \frac{3}{x+1}\right]_1^2$ $= 2\ln\left\lvert\frac{1}{\frac{1}{2}}\right\rvert + 3\left(\frac{1}{3} - \frac{1}{2}\right)$ $= 2ln2 - \frac{1}{2}$ $\quad or \quad$ $\ln 4 - \frac{1}{2}$	1 mark: correctly integrates 1 mark: correct evaluation (must give the answer in the simplest form	
d)	$\lvert z-1\rvert \leq 1$ *and* $\lvert z-2\rvert = 1.$ $\lvert z-1\rvert \leq 1$ is the circle and the interior of circle with centre (1, 0) and radius 1. $\lvert z-2\rvert = 1$ circle with centre (2,0) and radius 1. Both these conditions are satisfied by arc DAC.	2 marks: Correct answer from correct working. 1 mark: Minor error or explains arc DAC satisfies the two conditions	

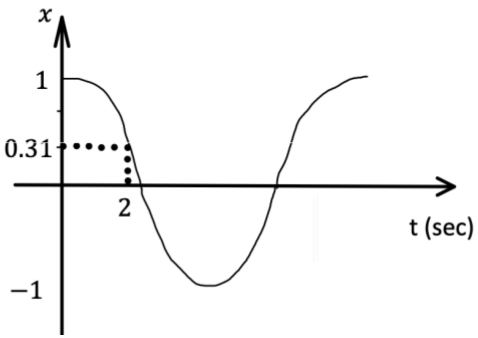
	 $OA \leq \lvert z \rvert \leq OC.\ or\ \ OD$ *As* $\angle OCB = \frac{\pi}{2}$, $\Rightarrow \quad OC^2 = OB^2 - BC^2 =. 4 - 1 = 3$ *Hence,* $\lvert z \rvert^2 \leq 3$	1 mark: uses the triangular inequality to obtain the inequation 1 mark: solves the equation and gives the greatest value	

Question 13

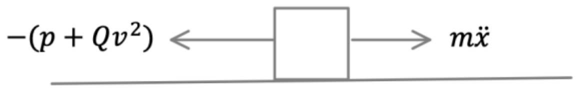
a)	$$p(n):\ n! < \left(\frac{n}{2}\right)^n, n > 5.\ n \in \mathcal{N}$$ Step 1: n = 6 $$6! < \left(\frac{6}{2}\right)^6 \Longrightarrow 6! < 3^6 \Longrightarrow 720 < 729$$ True $\therefore p(6)\ \ is\ true.$ 1 mark Step 2 Suppose p(k) is true. $p(k):\ k! < \left(\frac{k}{2}\right)^k \quad , \quad k > 5 \qquad (1)$ Proving p(k+1) is true. Multiplying (1) by (k+1) $$p(k+1):\ k!\,(k+1) < \left(\frac{k}{2}\right)^k (k+1)$$ $$(k+1)! < \frac{k^k}{2^k}(k+1)$$ $$(k+1)! < \frac{k^k(k+1) \times 2}{2^k \times 2} \qquad (2) \qquad 1\ mark$$ Consider $\left(1+\frac{1}{k}\right)^k$ $$\left(1+\frac{1}{k}\right)^k = 1 + k\left(\frac{1}{k}\right) + \left(\frac{k}{2}\right)\left(\frac{1}{k}\right)^2 + \cdots$$ Hence, $\left(1+\frac{1}{k}\right)^k > 2$ Hence, from (2) $$(k+1)! < \frac{k^k(k+1) \times 2}{2^k \times 2} < \frac{k^k(k+1)}{2^{k+1}}\left(1+\frac{1}{k}\right)^k$$	1 mark: Proves the basic result 1 mark: writes the p(k) and p(k+1) statements and demonstrates the basic manipulation 2 mark: proves $\left(1+\frac{1}{k}\right)^k > 2$ and proves p(k+1) is true and gives a concluding statement 1 mark: attempts and reasonable manipulation to prove the inequality and gives a concluding statement	

	$\therefore\ (k+1)! < \frac{k^k(k+1)}{2^{k+1}} \times \frac{(k+1)^k}{k^k}$ $\therefore\quad (k+1)! < \frac{(k+1)^{k+1}}{2^{k+1}}$ Hence, p(k+1) is true. Thus p(k) is true $\Longrightarrow p(k+1)\ is\ true\ for\ \forall k > 5,\ \ k \in \mathcal{N}$ $\therefore$ by principle of mathematical induction, $p(n)\ is$ true $for\ \forall n > 5,\ \ n \in \mathcal{N}$		
(b)(i)	$I_n = \int x^n e^{ax} dx$ Let $u = x^n,\ \ u' = nx^{n-1}$ $v' = e^{ax} \qquad v = \frac{e^{ax}}{a}$ $I_n = \frac{x^n}{a}e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx$ $I_n = \frac{x^n}{a}e^{ax} - \frac{n}{a}I_{n-1}$ as required.	1 mark: gives the correct expressions for u, u', v, v' or significant progress towards solution 1 mark: gives correct steps to conclusion	
b(ii)	$I_3 = \int_0^1 x^3 e^{2x} dx$ $I_3 = \left[\frac{x^3}{2}e^{2x}\right]_0^1 - \frac{3}{2}I_2 = \frac{e^2}{2} - \frac{3}{2}I_2$ $I_2 = \left[\frac{x^2}{2}e^{2x}\right]_0^1 - \frac{2}{2}I_1 = \frac{e^2}{2} - I_1$ $I_1 = \left[\frac{x}{2}e^{2x}\right]_0^1 - \frac{1}{2}I_0 = \frac{e^2}{2} - \frac{1}{2}I_0$ $I_0 = \int_0^1 e^{2x}dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{e^2}{2} - \frac{1}{2}$	1 mark: some progress towards solution. Eg. One correct step say fromI_3 to I_2. 1 mark: gives correct expression for I_0 1 mark: gives correct solution	

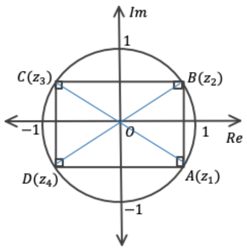
	$I_1 = \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right) = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$ $I_2 = \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right) = \frac{e^2}{2} - \left(\frac{e^2}{4} + \frac{1}{4}\right) = \frac{e^2}{4} - \frac{1}{4}$ $I_3 = \frac{e^2}{2} - \frac{3}{2}I_2 = \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2}{4} - \frac{1}{4}\right) = \frac{e^2}{8} + \frac{3}{8}$		
(c)(i)	$x = r\cos(\omega t + \phi)$ $\dot{x} = -r\omega\ \sin(\omega t + \phi)$ $\ddot{x} = \ -r\omega^2 \sin(\omega t + \phi)$ $\ddot{x} = \ -\omega^2\ x$ Ie. $r\cos(\omega t + \phi)$ satisfies the equation $\ddot{x} = \ -\omega^2\ x$.	1 mark: proves the result with all lines of working	
C(ii)	Crest to trough is 2m $\therefore$ amplitude $a = 1\ m$ We are considering the naval vessel while it is on the crest and the bullet will hit it provided it stayed stationary.  To travel a distance of 2000 m, the bullet takes $\frac{2000\ m}{1000 m/s}$ = 2 sec We will calculate the vertical height (x) of the vessel after 2 seconds. Period $T = \frac{2\pi}{n}$	1 mark: calculates the time taken for a distance of 2000m. t = 2 2 marks: calculates $a, n\ and\ \alpha$ and writes the equation of motion of the boat. 1 mark: minor error in calculating $a, n\ and\ \alpha$ and writes the equation of motion of the boat.	

	$10 = \frac{2\pi}{n} \Longrightarrow \quad n = \frac{2\pi}{10} = \frac{\pi}{5}$ $x = a\cos(nt + \alpha)$ $x = 1\cos(nt + \alpha)$ At $t = 0, x = 1 \quad (at\ the\ crest)$ $\therefore cos\alpha = 1 \qquad ie.\alpha = 0$ Hence, $x = 1\cos\left(\frac{\pi}{5}t\right)$ When t = 2, $x = \cos\frac{2\pi}{5} = \quad 0.309 \ldots..m$  Ie. the difference in height of the bullet and the target is $1\ m - 0.309 \ldots m$ $= 0.6909 \ldots.$ $\approx \quad 0.69\ m$	1 mark: calculates the displacement of the boat at t = 2 and calculates the difference in displacements of the bullet and the boat.	

Question 14

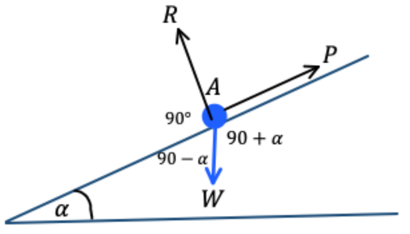
14(a) (i)	 $m\ddot{x} = -(P + Qv^2)$ $\ddot{x} = -\frac{1}{m}(P + Qv^2)$	1 mark: correct free body diagram, equation of motion and derives for $\ddot{x}$	
(ii)	$\ddot{x} = -\frac{1}{m}(P + Qv^2)$ If $P = 0$, $\ddot{x} = -\frac{Q}{m}v^2$ $\ddot{x} = -\frac{Q}{m}v^2$ $\ddot{x} = -\frac{Q}{m}v^2$ $v\frac{dv}{dx} = -\frac{Q}{m}v^2$ $Hence, \frac{dx}{dv} = -\frac{m}{Qv}$ $when\ x = 0, v = \frac{3U}{2}$ and when $x = D, v = U$ $Hence, \int_0^D dx = -\frac{m}{Q}\int_{\frac{3U}{2}}^{U} \frac{1}{v}dv$ $D = -\frac{m}{Q}\ln\left\|\frac{U}{\frac{3U}{2}}\right\|$ $D = -\frac{m}{Q}\ln\left\|\frac{2}{3}\right\| = \frac{m}{Q}\ln\left\|\frac{3}{2}\right\|$	1 mark: expresses $\ddot{x} = v\frac{dv}{dx}$ and writes the equation 1 mark: separates the differential equation into variable separable form 1 mark: Integrates, evaluates and simplifies	

(iii)	If $P>0$, then $\frac{dv}{dx} = -\left(\frac{P+Qv^2}{mv}\right)$ Hence, $\frac{dx}{dv} = -\frac{mv}{P+Qv^2}$ Set the distance at which $V = U$ as $x = 0$, and $V = 0$, when, say $x = D$ $$\frac{dx}{dv} = -\frac{mv}{P+Qv^2} \qquad 1\ mark$$ $\int_0^D dx = \int_U^0 -\frac{mv}{P+Qv^2}\,dv$ $D = -\frac{m}{2Q}\int_U^0 \frac{2Qv}{P+Qv^2}\,dv$ $D = -\frac{m}{2Q}[\ln\lvert P+Qv^2\rvert]_U^0 = -\frac{m}{2Q}\ln\left\lvert\frac{P}{P+QU^2}\right\rvert \quad 1\ mark$ $= \frac{m}{2Q}\ln\left\lvert\frac{P+QU^2}{P}\right\rvert$ $= \frac{m}{2Q}\ln\left\lvert 1+\frac{Q}{P}U^2\right\rvert \equiv \lambda\ln(1+kU^2)$ Hence, $\lambda = \frac{m}{2Q}$ and $k = \frac{Q}{P} \quad 1\ mark$	1 mark: gives the correct differential equation 1 mark: Integrates the differential equation correctly 1 mark: correctly simplifies to give the values of λ and k.	
14 (b)(i)	We know, $\lvert z_1 - z_2\rvert^2 = (z_1 - z_2)(\overline{z_1 - z_2})$ $= (z_1 - z_2)(\overline{z_1} - \overline{z_2})$ $= z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} = \lvert z_1\rvert^2 + \lvert z_2\rvert^2 - z_1\overline{z_2} - z_2\overline{z_1}$ Let E = $\lvert z_1 - z_2\rvert^2 + \lvert z_2 - z_3\rvert^2 + \lvert z_3 - z_4\rvert^2 + \lvert z_4 - z_1\rvert^2$	2 marks: Correct proof and reasoning 1 mark: expresses $\lvert z\rvert^2$ $as\ z\bar{z}$ and uses	

	$E = 2(\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \|z_4\|^2) - \overline{z_1}(z_2 + z_4) - \overline{z_2}(z_1 + z_3) - \overline{z_3}(z_2 + z_4) - \overline{z_4}(z_1 + z_3)$ $= 2(\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \|z_4\|^2) - (z_2 + z_4)(\overline{z_1} + \overline{z_3}) - (z_1 + z_3)(\overline{z_2} + \overline{z_4})$ However, $z_2 + z_4 = -(z_1 + z_3)$ as $z_1 + z_2 + z_3 + z_4 = 0$. $= 2(\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \|z_4\|^2) + (z_1 + z_3)(\overline{z_1} + \overline{z_3}) + (z_2 + z_4)(\overline{z_2} + \overline{z_4})$ Hence, $\|z_1 - z_2\|^2 + \|z_2 - z_3\|^2 + \|z_3 - z_4\|^2 + \|z_4 - z_1\|^2 =$ $= 2(\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \|z_4\|^2) + \|z_1 + z_3\|^2 + \|z_2 + z_4\|^2$ (statement 1) $= 2 \times 4 + \|z_1 + z_3\|^2 + \|z_2 + z_4\|^2$ Hence, $\|z_1 - z_2\|^2 + \|z_2 - z_3\|^2 + \|z_3 - z_4\|^2 + \|z_4 - z_1\|^2 \geq 8$	$\|z_1\| = \|z_2\| = \|z_3\| = \|z_4\| = 1$	
b(ii) α	From (i), using statement 1, $E - 2(\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \|z_4\|^2)$ $= \|z_1 + z_3\|^2 + \|z_2 + z_4\|^2 = 0$ $\Leftrightarrow z_1 + z_3 = 0$ and $z_2 + z_4 = 0$.	1 mark: proves the result	
β	 The sum of squares of the side lengths of a rectangle with vertices of the form z = $\pm a \pm bi$ equals 8	1 mark: for correct diagram 1 mark: for the geometrical reasoning	
(iii)	$\omega + \overline{\omega} = 2\,Re(\omega)$ $\omega + \overline{\omega} = \frac{u - v}{1 + uv} + \overline{\left(\frac{u - v}{1 + uv}\right)}$ $= \frac{u-v}{1+uv} + \frac{\overline{u}-\overline{v}}{1+\overline{u}\overline{v}}$	2 mark: correct solution from correct working 1 mark: explains $\omega + \overline{\omega} = 2\,Re(\omega)$ and attempts to simplify the expression	

	We expand and simplify: $= \frac{u-v}{1+uv} + \frac{\bar{u}-\bar{v}}{1+\bar{u}\bar{v}}$ $= \frac{u-v+u\bar{u}\bar{v}-\bar{u}v\bar{v}+\bar{u}-\bar{v}+u\bar{u}v-uv\bar{v}}{(1+uv)(1-\bar{u}\bar{v})}$ $u\bar{u} = 1$ and $v\bar{v} = 1$ $\therefore \frac{u-v+u\bar{u}\bar{v}-\bar{u}v\bar{v}+\bar{u}-\bar{v}+u\bar{u}v-uv\bar{v}}{(1+uv)(1-\bar{u}\bar{v})} = 0$ $= \frac{u-v}{1+uv} + \frac{v-u}{1+uv} = 0$ $ie. \quad 2\,Re(\omega) = 0$ $\therefore Re(\omega) = 0$		
(iv)	Let $u = r_1e^{i\theta}$ and v= $r_2e^{i\alpha}$, where $r_1 < 1, r_2 < 1$ $1 - \bar{u}v = 1 - r_1r_2e^{i(\alpha-\theta)}$ $\lvert 1-\bar{u}v\rvert^2 = (1 - r_1r_2\cos(\alpha-\theta))^2 + (r_1r_2)^2\sin^2(\alpha-\theta)$ $= 1 - 2r_1r_2\cos(\alpha-\theta) + r_1^2r_2^2$ $\leq (1-r_1r_2)^2$ as $-1 \leq \cos(\alpha-\theta) \leq 1$ $= (1 - \lvert u\rvert\lvert v\rvert)^2$ That is $\lvert 1-\bar{u}v\rvert^2 \leq (1-\lvert u\rvert\lvert v\rvert)^2$ Hence, $\lvert 1-\bar{u}v\rvert \leq 1 - \lvert u\rvert\lvert v\rvert$ $\frac{1}{1-\lvert u\rvert\lvert v\rvert} \leq \frac{1}{\lvert 1-\bar{u}v\rvert}$ (1) 1 mark (result 1) Also, using triangular inequalities, $\big\lvert \lvert u\rvert - \lvert v\rvert \big\rvert \leq \lvert u-v\rvert$ (2) 1 mark and conclusion From (1) and (2), $\frac{\big\lvert \lvert u\rvert - \lvert v\rvert \big\rvert}{1-\lvert u\rvert\lvert v\rvert} \leq \left\lvert \frac{u-v}{1+\bar{u}\,v} \right\rvert$	1 mark: expresses u and v correctly and writes an expression for $\lvert 1-\bar{u}v\rvert^2$ 1 mark: proves the inequality result 1 1 mark: Uses triangular inequality and proves the result	

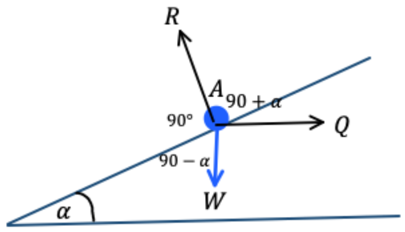
Question 15

a)(i)	$\frac{1}{k-\frac{1}{2}}-\frac{1}{k+\frac{1}{2}}=\frac{k+\frac{1}{2}-\left(k-\frac{1}{2}\right)}{k^2-\frac{1}{4}}$ $=\frac{1}{k^2-\frac{1}{4}}>\frac{1}{k^2}$ $Hence, \frac{1}{k^2}<\frac{1}{k-\frac{1}{2}}-\frac{1}{k+\frac{1}{2}}$	2 marks: correct proof 1 mark: algebraic manipulation and some relevant logic of inequality demonstrated	
a)(ii)	From (i), $\sum_{k=1}^{n}\frac{1}{k^2}<\sum_{k=1}^{n}\frac{1}{k-\frac{1}{2}}-\frac{1}{k+\frac{1}{2}} \quad 1\ mark$ $<\left(\frac{1}{\frac{1}{2}}-\frac{1}{\frac{3}{2}}\right)+\left(\frac{1}{\frac{3}{2}}-\frac{1}{\frac{5}{2}}\right)+\cdots.+\left(\frac{1}{n-\frac{1}{2}}\right)-\left(\frac{1}{n+\frac{1}{2}}\right)$ $<\frac{1}{\frac{1}{2}}-\frac{1}{n+\frac{1}{2}} \quad 1\ mark$ $<\frac{2}{1}-\frac{1}{n+\frac{1}{2}}$ $<2 \quad as\ \frac{1}{n+\frac{1}{2}}>0\ \forall n\in\mathcal{Z}^+$ 1 mark	1 mark: using result (i), writes the inequality of the sum of the series 1 mark: simplifies the series Proves the result with reasoning and appropriate mathematical notations	
b)		2 marks: correct answer from correct working	

<table>
<tr>
<td></td>
<td>Using Lami's theorem,

$$\frac{P}{\sin(180-\alpha)} = \frac{W}{sin90} = \frac{R}{\sin(90+\alpha)}$$

Hence, $P = W\sin\alpha$



$$\frac{Q}{\sin(180-\alpha)} = \frac{W}{\sin(90+\alpha)} = \frac{R}{\sin\ 90}$$

Hence, $Q = W\tan\alpha$

$$\frac{1}{P^2} - \frac{1}{Q^2} = \frac{1}{W^2}(cosec^2\alpha - \cot^2\alpha)$$

$$\frac{1}{P^2} - \frac{1}{Q^2} = \frac{1}{W^2}$$</td>
<td>1 mark: Draws the free body diagram, writes the force equations</td>
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<td>c)</td>
<td>Let A be the point $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $P\begin{pmatrix}1\\2\\0\end{pmatrix}$

$\overrightarrow{AP} = \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\2\\-1\end{pmatrix}$

$\underset{\sim}{v} = \begin{pmatrix}0\\1\\1\end{pmatrix}$</td>
<td>1 mark: calculates $\overrightarrow{AP}$, $\overrightarrow{AP}.v$ correctly

1 mark: correctly calculates the $Proj_v\ \overrightarrow{AP}$

1 mark: calculates the shortest distance</td>
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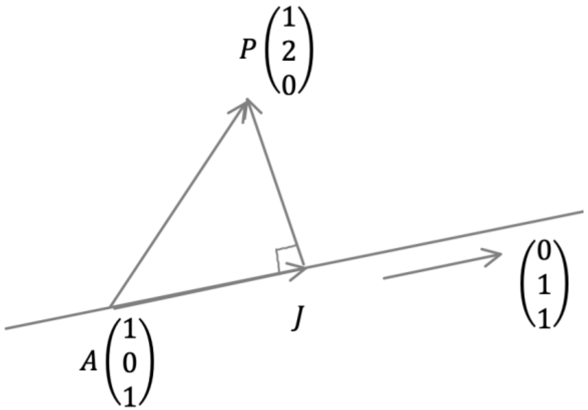


$$Proj_v\,\overrightarrow{AP} = \frac{\overrightarrow{AP}.v}{v.v}v$$

$$\overrightarrow{AP}.v = \begin{pmatrix}0\\2\\-1\end{pmatrix}.\begin{pmatrix}0\\1\\1\end{pmatrix} = 0+2-1 = 1 \quad \boldsymbol{1\,mark}$$

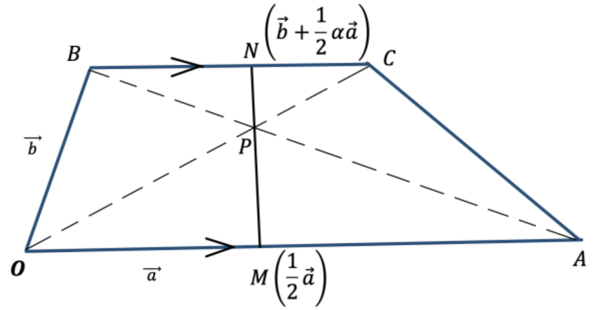
$$v.v = \begin{pmatrix}0\\1\\1\end{pmatrix}.\begin{pmatrix}0\\1\\1\end{pmatrix} = 0+1+1 = 2$$

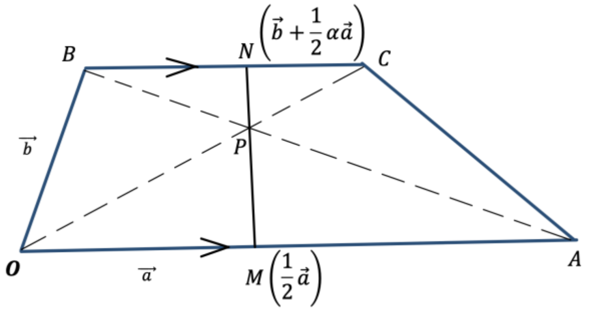
$$Proj_v\,\overrightarrow{AP} = \frac{\overrightarrow{AP}.v}{v.v}v = \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

The shortest distance =

$$= \left|\overrightarrow{AP} - Proj_v\,\overrightarrow{AP}\right| = \left|\begin{pmatrix}0\\2\\-1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}\right|$$

$$= \frac{3}{2}\left|\begin{pmatrix}0\\1\\-1\end{pmatrix}\right| = \frac{3}{\sqrt{2}}$$

	Method 2 $A\begin{pmatrix}0\\1\\1\end{pmatrix}$, $J\begin{pmatrix}1\\\lambda\\1+\lambda\end{pmatrix}$ $\overrightarrow{PJ}=\begin{pmatrix}1-1\\\lambda-2\\1+\lambda-0\end{pmatrix}=\begin{pmatrix}0\\\lambda-2\\1+\lambda\end{pmatrix}$ $\begin{pmatrix}0\\\lambda-2\\1+\lambda\end{pmatrix}.\begin{pmatrix}0\\1\\1\end{pmatrix}=0.$ (perpendicular) $\lambda=\frac{1}{2}$ $\therefore\ \overrightarrow{PJ}=\begin{pmatrix}0\\-3/2\\3/2\end{pmatrix}$ $\lvert\overrightarrow{PJ}\rvert=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2}=\frac{3}{\sqrt{2}}$		
d) (i)	 Since, $BC \parallel OA$, $\overrightarrow{BC}=\alpha\,\overrightarrow{OA}=\alpha\,\overrightarrow{a}$ for some constant α. Equation of OC is $\overrightarrow{r_1}=t\,(b+\alpha\,\overrightarrow{a}\,)$	1 mark: Explains that $\overrightarrow{BC}=\alpha\,\overrightarrow{OA}$ and writes the vector representations for OC	

(ii)	 Similarly, equation of AB is $\overrightarrow{r_2}=\overrightarrow{a}+\lambda\,(\overrightarrow{b}-\overrightarrow{a})$ Let P be the point of intersection of OC and AB. Then, at point P, $t\,(b+\alpha\,\overrightarrow{a})=\overrightarrow{a}+\lambda\left(\overrightarrow{b}-\overrightarrow{a}\right)$ for some value of $t\ and\ \lambda$. $\Longrightarrow$ $(t\alpha-1+\lambda)\overrightarrow{a}\quad=(\lambda-t)\overrightarrow{b}$ Since, $\overrightarrow{a}$ and $\overrightarrow{b}$ are non-parallel vectors, we must have $t\alpha-1+\lambda=0$ and $\lambda-t=0$ $\lambda=t$ $\therefore\ \ t\alpha-1+t=0$ $t=\dfrac{1}{\alpha+1}$ Hence, the position vector of P is: $\overrightarrow{r_3}=\dfrac{1}{\alpha+1}\left(\vec{b}+\alpha\,\overrightarrow{a}\right)\ \boldsymbol{1\ mark}$ Equation of MN: $r_4=\frac{1}{2}\overrightarrow{a}+k\,\overrightarrow{MN}$ $=\frac{1}{2}\overrightarrow{a}+k\,(\overrightarrow{b}+\frac{1}{2}\alpha\,\overrightarrow{a}-\frac{1}{2}\overrightarrow{a})$	1 mark: Equates the equations for OC and AB to help find B and gives $t=\frac{1}{\alpha+1}$ 1 mark: compare coefficients and finds the position vector of P 1 mark: writes the vector equation of MN 1 mark: Proves that P lies on MN.	

	$=\frac{1}{2}\overrightarrow{a}+k\left(\overrightarrow{b}+\frac{1}{2}(\alpha-1)\overrightarrow{a}\right)$ **1 mark** Note: Comparing r_3 and r_4, we note that the coefficient $\overrightarrow{b}$ is $\frac{1}{\alpha+1}$ Hence, k must be $\frac{1}{\alpha+1}$ $r_4=\frac{1}{2}\overrightarrow{a}+\frac{1}{\alpha+1}\left(\overrightarrow{b}+\frac{1}{2}(\alpha-1)\overrightarrow{a}\right)$ $=\frac{1}{\alpha+1}\vec{b}+\frac{\alpha-1}{2(\alpha+1)}\vec{a}+\frac{1}{2}\vec{a}$ $=\frac{1}{\alpha+1}\vec{b}+\frac{1}{2(\alpha+1)}(\alpha-1+\alpha+1)\vec{a}$ $=\frac{1}{\alpha+1}\left(\vec{b}+\alpha\overrightarrow{a}\right)$ which is the position vector of P. That is P lies on MN **1 Mark**		

Question 16

a)	Suspose c is a real number such that $r_3x^3 + r_2x^2 + r_1x + r_0 = 0$ where, r_0, r_1, r_2 and r_3 are rational numbers. By the definition of rational, $r_0 = \frac{a_0}{b_0}$, $r_1 = \frac{a_1}{b_1}, r_2 = \frac{a_2}{b_2}$ and $r_3 = \frac{a_3}{b_3}$ for some integers **1 mark** a_0, a_1, a_2 and $a_3 \in \mathcal{N}$ and non-zero integers b_0, b_1, b_2 and $b_3 \in \mathcal{Z}^+$. **1 mark** By substitution, $r_3c^3 + r_2c^2 + r_1c + r_0$ $= \frac{a_3}{b_3}c^3 + \frac{a_2}{b_2}c^2 + \frac{a_1}{b_1}c + \frac{a_0}{b_0}$ $= \frac{a_3b_2b_1b_0}{b_3b_2b_1b_0}c^3 + \frac{a_2b_3b_1b_0}{b_3b_2b_1b_0}c^2 + \frac{a_1b_3b_2b_0}{b_3b_2b_1b_0}c + \frac{a_0b_3b_2b_1}{b_3b_2b_1b_0} = 0$ $b_3b_2b_1b_0 \neq 0$, by definition. **1 mark** Multiplying by $b_3b_2b_1b_0$ gives $a_3b_2b_1b_0 \,.\, c^3 + a_2b_3b_1b_0 \,.\, c^2 + a_1\, b_3b_2b_0 \,.\, c + a_0b_3b_2b_1 = 0$ Let $n_3 = a_3b_2b_1b_0$, $n_2 = a_2b_3b_1b_0$, $n_1 = a_1b_3b_2b_0$ and $n_0 = a_0\, b_3b_2b_1$ where n_3, n_2, n_1 and n_0 are all integers, being product of integers. Thus, c satisfies a polynomial with integer coefficients.	1 mark: expresses r_0, r_1, r_2 and r_3 as rational numbers 1 mark: clearly explains the domain of numerator and denominator of the rational coefficients. 2 marks: correct proof • Clear reasoning • Clearly demonstrates how the integer coefficients are derived 1 mark: attempts to prove that the coefficients are integers, however, gives clear reasoning for each assertion.	
(b)	$P\big(x(\theta)\big) = h + x(\theta)tan\theta - \frac{g(x(\theta))^2}{2v^2}\sec^2\theta$		

	We are looking for the maximum value of $x(\theta)$, hence $x'(\theta) = 0$ $P'\big(x(\theta)\big).x'(\theta)=$ $x(\theta)\sec^2\theta + x'(\theta)tan\theta$ $-\frac{g}{2v^2}[\,\big(x(\theta)\big)^2 \times 2sec\theta.sec\theta tan\theta + 2x(\theta)x'(\theta)\sec^2\theta]$ Put $x'(\theta) = 0$ $0 = x(\theta)\sec^2\theta - \frac{g}{2v^2}\big(x(\theta)\big)^2 \sec^2\theta\, tan\theta$ $\theta \neq \frac{\pi}{2},\;\; x(\theta) \neq 0$ Hence, $$x(\theta)\sec^2\theta = \frac{g}{v^2}\big(x(\theta)\big)^2 2\sec^2\theta\, tan\theta$$ $\frac{g}{2v^2}x(\theta)tan\theta = 1$ $x(\theta) = \frac{v^2}{g}cot\theta$ The given function is a concave down quadratic in $x(\theta)$. Hence, $x(\theta)$ will be maximum, when $\theta = \theta_m$ $x_m = \frac{v^2}{g}cot\theta_m$	1 mark: differentiates the compound function 1 mark: sets $x'(\theta) = 0$ and solves for $x(\theta)$ to prove the result	
c(i)	$x = vcos\theta\; t$ $\therefore\;\; t = \frac{x}{vcos\theta}$ $y = h + v\, sin\theta t - \frac{1}{2}gt^2$ $y = h + v\, sin\theta.\frac{x}{vcos\theta} - \frac{1}{2}g\left(\frac{x}{vcos\theta}\right)^2$	1 mark: eliminates the parameter θ from the two simultaneous equations of motion	

	$\therefore \quad y = \quad h + x\tan\theta - \dfrac{gx^2}{2v^2}\sec^2\theta$		
c(ii)	An enveloping parabola can be got by maximizing the height of the projectile for a given horizontal distance x, which will give us the path that encloses all possible paths. $y \; = h + xtan\theta - \dfrac{gx^2}{2v^2}\sec^2\theta$ Let $u = tan\theta$ $y \; = h + ux - \dfrac{gx^2}{2v^2}(1+u^2)$ $\dfrac{dy}{du} = x - \dfrac{gx^2}{2v^2} \times 2u$ $\dfrac{dy}{du} = 0$ $x \neq 0$ $x = \dfrac{gx^2}{2v^2} \times 2u \Longrightarrow u = \dfrac{v^2}{gx}$ **1 mark** Hence, the envelope is $y_{max} = h + \left(\dfrac{v^2}{gx}\right)x - \dfrac{gx^2}{2v^2}\left(1 + \left(\dfrac{v^2}{gx}\right)^2\right)$ $= \quad h + \dfrac{v^2}{g} - \dfrac{gx^2}{2v^2} - \dfrac{gx^2}{2v^2} \times \dfrac{v^4}{g^2x^2}$ $= \quad h + \dfrac{v^2}{g} - \dfrac{gx^2}{2v^2} - \dfrac{gx^2}{2v^2} \times \dfrac{v^4}{g^2x^2}$ Hence, the equation of the envelope is $\phi(x) \; = \; h + \dfrac{v^2}{2g} - \dfrac{gx^2}{2v^2}$	1 mark: Differentiates $y \; = h + ux - \frac{gx^2}{2v^2}(1+u^2)$ and finds $tan\theta$ that maximises y. 1 mark: substitutes u and gets the equation of the envelope.	

c(iii)	At the point $x = c$, $P(x)$ and $\phi(x)$ intersect the plane $y = mx$. $h + \frac{v^2}{2g} - \frac{gc^2}{2v^2} = mc$ $\frac{gc^2}{2v^2} + mc - \left(\frac{2gh + v^2}{2g}\right) = 0$ $c^2 + \frac{2v^2}{g} mc - \frac{2v^2}{g}\left(\frac{2gh + v^2}{2g}\right) = 0$ $c^2 + \frac{2mv^2}{g} c - \frac{v^2}{g^2}(2gh + v^2) = 0$ **1 mark** Hence, the value of c is $c = \frac{-\frac{2mv^2}{g} \pm \sqrt{\frac{4m^2v^4}{g^2} + \frac{4v^2}{g^2}(2gh + v^2)}}{2}$ $c = -\frac{mv^2}{g} \pm \frac{v^2}{g}\sqrt{m^2 + \frac{2gh + v^2}{v^2}}$ $c = -\frac{mv^2}{g} \pm \frac{v^2}{g}\sqrt{m^2 + \frac{2gh}{v^2} + 1}$ **1 mark**	2 marks: Equates the two functions and forms the simultaneous equation 1 mark: correctly solves to give the value of c, the x-coordinate of the point of impact.	
c(iv)	At the point $x = c$, $P(x)$ and $\phi(x)$ intersect the plane $y = mx$._ That is $c = x_m$.	3 marks: refers to the solution from (b), sets $c = x_m$ and uses the result in (ii) to prove the result, with reasoning	

	From (b), the optimal launching angle θ_m that gives the maximum range x_m is: $x_m = \frac{v^2}{g}\cot\theta_m$ $\quad$ $\boldsymbol{1\ mark}$ Hence, from (ii), $c = x_m$ $-\frac{mv^2}{g} \pm \frac{v^2}{g}\sqrt{m^2 + \frac{2gh}{v^2} + 1} = \frac{v^2}{g}\cot\theta_m$ $\cot\theta_m = \frac{g}{v^2}\left(-\frac{mv^2}{g} \pm \frac{v^2}{g}\sqrt{m^2 + \frac{2gh}{v^2} + 1}\right)$ $cot\theta_m = -m \pm \sqrt{m^2 + \frac{2gh}{v^2} + 1}$ Optimal angle of projection θ_m $\theta_m = \cot^{-1}\left(-m \pm \sqrt{m^2 + \frac{2gh}{v^2} + 1}\right)$ However, $0 < \theta < \frac{\pi}{2}$ Hence, $\cot\theta_m > 0.$ and reject	2 marks: refers to the solution from (c), sets $c = x_m$ and uses the result in (ii) to prove the result 1 mark: refers to the result in (b) and identifies $c = x_m$ and attempts to solve	

	$\cot\theta_{m} = -m - \sqrt{m^2 + \frac{2gh}{v^2} + 1} < 0$ Hence, $\cot\theta_{m} = -m + \sqrt{m^2 + \frac{2gh}{v^2} + 1}$		